Interference of Bose condensates

M. Naraschewski,* H. Wallis, and A. Schenzle*
Max–Planck–Institut für Quantenoptik
Hans–Kopfermann–Straße 1, D–85748 Garching, Germany

J. I. Cirac
Departamento de Fisica Aplicada, Universidad de Castilla-La Mancha,
13071 Ciudad Real, Spain

P. Zoller
Institut für theoretische Physik, Universität Innsbruck, Technikerstr. 25, Austria
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Abstract

We investigate the prospects of atomic interference using samples of Bose condensed atoms. First we show the ability of two independent Bose condensates to create an interference pattern. This holds even if both condensates are described by Fock states. Thus, the existence of an experimental signature for a broken gauge symmetry, seen in a single run of the experiment, is not necessarily reflected by a broken symmetry on the level of the quantum mechanical state vector. Based on these results, we simulate numerically a recent experiment with two independent Bose condensates [K.B. Davis et al., PRL 75, 3969 (1995)]. The existence of interference fringes is predicted based on the nonlinear Schrödinger equation. Finally we study theoretically the influence of finite temperatures on the visibility of the interference in a double pinhole configuration.

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I. INTRODUCTION

Recent realizations of Bose-Einstein condensation (BEC) in dilute and ultracold gases [1,2,3] have attracted vivid interest. It is hoped that the study of those experimental systems will give new insight into the physics of BEC. Since the current understanding of BEC is largely influenced by the concept of a macroscopic wave function, the study of this feature is of foremost importance. The investigation of interference phenomena should be perfectly apted for this purpose. Another motivation for the study of interference properties is the envisioned development of a new source of atoms, based on BEC, with high flux and coherence, that is expected to stimulate atomic interference experiments.

Although interference is a well known phenomenon in quantum statistically degenerate systems (cf. Josephson effect), surprisingly little quantitative analysis has been performed up to now, to investigate the interference properties of a Bose condensate. In this paper we study theoretically two interference experiments that could be performed with the technology of current BEC experiments [1,2,3].

In Sec. II we focus on the general possibility of interference between two independent Bose condensates. This topic is closely related to earlier studies, showing interference between two independent laser beams [4,5]. Interference of atoms, originating from independent sources, differs qualitatively from usual interference experiments, where atoms interfere only with themselves. In this case the phase of the interference pattern is uniquely defined by the geometry of the setup. For two independent condensates, however, an interference pattern will be exhibited with a phase that unpredictably varies between *different runs* of the experiment. This implies that the system possesses some symmetry that is spontaneously broken in a single run of the experiment. An intuitive explanation for this interference follows from the common notion of spontaneously broken gauge symmetry, implying the existence of a macroscopic wave function. We show, however, that the system behaves identically under circumstances where the broken symmetry is not reflected by the quantum mechanical state vector, e.g. if the two condensates are initially in Fock states. In this case, it is still possible to derive from quantum mechanics the existence of interference patterns in a single run.

Having shown the interference of two independent Bose condensates in quite general terms, we turn more realistic in Sec. III. This includes accounting for the finite size of experimental condensates and the important influence of atom interactions. Finite temperature effects, however, are still neglected. In principle, a setup for interference of two condensates has been realized in the BEC experiment of [3]. There, two condensates are stored in a magnetic trap, separated by a light beam. After releasing them from the trap, they expand and eventually overlap. However, no interference has been observed in the reported experiment [3]. We simulate this experiment numerically using the nonlinear Schrödinger equation. We are able to reproduce the observed final size of the atomic cloud in good agreement with [3]. In addition it is shown that interference fringes should exist with a spatial period smaller than the present experimental resolution.

In Sec. IV we investigate the more conventional case where a single condensate is released from a harmonic trap and subsequently interferes with itself in a simple double pinhole experiment. In this calculation we allow for finite temperatures but neglect atom interactions. It is investigated, to what extent the existence of interference fringes might serve as a signature of Bose condensation. It turns out that the appearance of a condensate is reflected

by an extremely sharp change in the visibility of the interference pattern at the critical temperature.

After completion of our work we have become aware of a recent paper by J. Javanainen and S. Yoo [6]. These authors have studied the interference of two independent Bose condensates with conclusions, similar to the ones reached in Sec. II, but based on a complementary derivation.

II. INTERFERENCE OF TWO INDEPENDENT BOSE CONDENSATES

A. Spontaneously broken gauge symmetry

In usual interference experiments (cf. Sec. IV), atomic beams are split apart in a suitable way, e.g. by a Young double slit, and recombined afterwards thereby leading to an interference signal. Since all atoms come from the same source and are not tracked on their way through the interferometer one usually notes that atoms interfere with themselves in such a situation. Here, we address the question whether an interference signal can be exhibited as well, when two *independent* Bose condensates merge.

Although, in this case atoms do not interfere with themselves but rather with other indistinguishable atoms, it should indeed be possible to observe such an interference under suitable conditions. A simple understanding of this phenomenon can be achieved using the notion of spontaneously broken gauge symmetry, that is widely regarded as a characteristic feature of BEC. We will shortly summarize this concept in the following.

A spontaneously broken symmetry, in general, implies that the behaviour of a *single* many-particle system differs from its *ensemble* average. The common paradigm for spontaneous symmetry breaking is a ferromagnet, where a single domain may show a magnetization while the mean magnetization, averaged over different domains, vanishes. It is a common approach in statistical physics [7] to investigate the existence of such a broken symmetry by adding a suitable small symmetry breaking field (Bogoliubov auxiliary field) to the Hamiltonian of the system. The ground state of this new Hamiltonian is meant to describe a particular single system at zero temperature with broken symmetry instead of the symmetric ensemble. The density matrix, describing the full ensemble, is then assumed to be a symmetry preserving, incoherent superposition of degenerate ground states, corresponding to different orientations of the Bogoliubov field.

So as to account for several observed interference phenomena like the Josephson effect, it is a common notion to postulate a spontaneously broken gauge symmetry [8] in the case of BEC. The corresponding Bogoliubov field consists in adding

$$\epsilon\Psi(x) + \epsilon^*\Psi^\dagger(x) \tag{2.1}$$

to the Hamiltonian, letting $\epsilon \to 0$ after having performed the thermodynamic limit. It does not correspond to any physical interaction, in contrast to the case of ferromagnetism. Its consequence is that a *single* Bose condensate is described by a *coherent state* with a distinguished phase. All measurable quantities of this particular single condensate are then fully characterized by the complex scalar field $\psi(x)$ with

$$\langle\Psi(x)\rangle_S = \psi(x). \tag{2.2}$$

Note, that the angular brackets in Eq. (2.2) refer only to the *subensemble* that is defined by the particular coherent state. The field $\psi(x)$ is usually called the *macroscopic wave function* of the condensate. The macroscopic wave function $\psi'(x)$ of any other condensate, seen in a different run of the same experimental setup, can be achieved by applying the gauge transformation

$$\Psi'(x) = \exp(i\alpha\hat{N})\Psi(x)\exp(-i\alpha\hat{N}) \tag{2.3}$$

with $\hat{N}$ being the number operator. This yields

$$\psi'(x) = e^{-i\alpha}\psi(x), \tag{2.4}$$

motivating the notion of a broken gauge symmetry. In contrast to the single system, the *full ensemble* does not show any phase dependence

$$\langle\Psi(x)\rangle = 0, \tag{2.5}$$

i.e. it is invariant under the symmetry transformation Eq. (2.3). The main advantage of this concept is that quantities, measured in a *single run* of the experiment, can still be expressed by simple quantum mechanical expectation values, while the true *ensemble* expectation values fail to characterize a single system properly.

So far, the analogy between spontaneous symmetry breaking in a ferromagnet and in BEC seems to hold perfectly. However, spontaneously broken gauge symmetry has to be taken with some caveat, as will be indicated in the following. In the case of ferromagnetism, the ground state of the unmodified Hamiltonian is *degenerate*. The effect of the Bogoliubov field consists only in selecting a particular one of these degenerate states for the characterization of a given domain.

Such a correspondence exists equally in the usual treatment of BEC. Here, the density matrix of the condensate at $T = 0$ is expressed in terms of *coherent states*

$$\rho_0 = \frac{1}{\pi}\int d^2\alpha\, \delta(|\alpha|^2 - N)\ |\alpha\rangle\langle\alpha|, \tag{2.6}$$

from which the Bogoliubov field selects a particular $|\bar{\alpha}\rangle$, due to the assumed broken gauge symmetry.

The justification of this treatment may be questioned, since the Bogoliubov field Eq. (2.1) does not correspond to a real physical field. In addition, atomic coherent states are no eigenstates of the particle number and violate atom number conservation, that is implied by fundamental superselection rules. It is therefore not obvious why a decomposition in terms of coherent states rather than in terms of number states is appropriate for the characterization of a single run of the experiment. Another subtle feature arises from the fact that the assumed degeneracy of the ground state requires the use of the grandcanonical ensemble with fluctuations of the particle number. In a microcanonical ensemble with a fixed particle number N, however, the ground state of the unmodified Hamiltonian may be *nondegenerate*, i.e. given by a single Fock state

$$\rho_0 = |N\rangle\langle N|. \tag{2.7}$$

Obviously, the concept of the Bogoliubov field, merely selecting one of several existing ground states, becomes meaningless in this case. This raises the important question whether a microcanonical description will also show BEC with all its characteristic features. A positive answer to this question would imply that the degeneracy of the ground state is not essential for the understanding of BEC, in contrast to the case of ferromagnetism. Indeed, a numerical simulation of the quantum kinetic equation [9] has shown that a macroscopic population of a nondegenerate ground state occurs in a system with a fixed particle number. It remains to be seen, whether the density matrix Eq. (2.7) is able to reproduce even those effects, that are commonly explained by a broken gauge symmetry.

We address this question in the following by investigating the interference properties of two independent Bose condensates. It will be shown, that the notion of a macroscopic wavefunction gives the proper intuitive understanding about a single run of such a fictitious experiment, no matter whether Eq. (2.6) or Eq. (2.7) is assumed. This indicates that a degeneracy of the ground state should not be essential for the explanation of the phenomena, that are usually regarded as an evidence for a spontaneously broken gauge symmetry.

B. Interference of two plane waves

We assume that two independent atomic beams, being Bose condensates, merge on a planar detection screen. The accumulation of a large number of particles on the screen during a given detection time τ_d will be regarded as a *single* run of the experiment. Only if we repeat the same experiment several times with a complete physical reset between different counting intervals, this will be called an *ensemble*. Since we assume a homogeneous velocity distribution of the incoming atoms, the distribution of particles, accumulated on the screen during τ_d in a *single* run is proportional to the line density $N(x)$ along the x-axis.

We describe the condensates by two independent macroscopically occupied momentum modes with x-momenta $\pm k_0$ and average occupation numbers $N_i = \left\langle a_i^\dagger a_i \right\rangle$ $(i = 1, 2)$. Their number fluctuations are assumed to be negligible in comparison with the mean occupation numbers:

$$\frac{\left\langle a_i^\dagger a_i^\dagger a_i a_i \right\rangle}{N_i N_i} = 1 + O\left(\frac{1}{N_i}\right). \tag{2.8}$$

This includes a representation of the condensates in terms of Eq. (2.6) or alternatively of Eq. (2.7).

The first step of our analysis is to determine the *ensemble* behaviour of the system. It is easily seen from

$$\langle N(x) \rangle \, V^{1/3} = N_1 + N_2 \tag{2.9}$$

that *no* interference is visible if the measurements of many runs are summed up. This result is no surprise due to the translational invariance of the system in x-direction. Usually, also a single run is sufficiently characterized by $\langle N(x) \rangle$. However, our considerations about broken gauge symmetry have already indicated that this may be wrong in the case of BEC. This implies that the density distribution $N(x)$, measured in a single run, might show a nontrivial spatial dependence. The interesting question is therefore, whether interference

exists in the counts from a *single* run of the experiment. A rigorous answer can be given by analyzing correlation functions or conditional probabilites like $\langle N(x)N(x+\Delta)\rangle$. If these show a nontrivial dependence on the relative distance Δ, it is clear that the individual density distribution $N(x)$ must be different from the ensemble average Eq. (2.9). However, before turning to the detailed discussion of correlation functions, we describe briefly the predictions of the broken gauge symmetry model for our situation.

1. Intuitive treatment

Here, we assume the notion of a broken gauge symmetry to be true. As a consequence, the macroscopic wave function of a single condensate would be a classical plane wave with an arbitrary but fixed phase factor. If we describe the full state vector by a product of two coherent states, corresponding to the two condensates, we get the combined macroscopic wave function

$$\psi(x)\sqrt{V^{1/3}} = e^{i\alpha_1}\sqrt{N_1}e^{ik_0x} + e^{i\alpha_2}\sqrt{N_2}e^{-ik_0x}, \tag{2.10}$$

which is a sum of the single macroscopic wave functions. It is obvious that this wave function leads to an interference pattern

$$N(x)V^{1/3} = (N_1 + N_2) + 2\sqrt{N_1N_2}\cos(2k_0x + \varphi) \tag{2.11}$$

on the screen which is determined by the *relative* phase $\varphi = \alpha_1 - \alpha_2$. This relative phase varies between different runs and thus leads to Eq. (2.9) in the ensemble. The unpredictable variation of φ between different runs would be usually interpreted as an experimental evidence for a broken gauge symmetry.

However, the intuitive treatment using Eq. (2.10) depends on assumptions, that were already questioned above. First, a degeneracy of the ground state is assumed, so as to decompose the density matrix into coherent states (cf. Eq. (2.6)). Then the single condensates are identified with specific *coherent* states with the intention that their expectation value $\left\langle \Psi^\dagger(x)\Psi(x)\right\rangle_S$ characterizes the density distribution of a single run. Regrettably, this procedure is not in rigorous agreement with atom number conservation. These demand that the condensate of a single run has a definite atom number, although we may not know it. This number is allowed to fluctuate only between different runs of the experiment. Taking number conservation serious, one might be misled to suggest that the density distribution of a single run is properly characterized by the expectation value $\left\langle \Psi^\dagger(x)\Psi(x)\right\rangle_S$ of *Fock* states. This would yield

$$N(x)V^{1/3} = N_1 + N_2 \tag{2.12}$$

even for a single run, which is in contradiction with Eq. (2.11).

It will be the result of the rest of this section to confirm the implications of Eq. (2.10) rigorously without using any of the mentioned assumptions. A counterintuitive implication of this derivation is that Eq. (2.11) instead of Eq. (2.12) gives the proper understanding of the single system behaviour even if the two condensates are described by Fock states. In addition, the following analysis illustrates, how one can extract detailed information about the behaviour of *single* systems from the quantum mechanical ensemble description.

Our aim, to characterize single runs of an experiment, confronts us with the problem that quantum mechanics by definition determines only properties of the ensemble, i.e. averages over many runs. Nevertheless, ensemble averages do allow to predict certain features of single runs which are not apparent in the ensemble by the analysis of correlation functions. Such an approach has found widespread use in quantum optics [10]. Characteristic features of the time evolution of a single atom could be derived in the case of the quantum jumps [11,12], eventually leading to the numerical tool of the Monte-Carlo wavefunction technique [13,14,15]. Here, we consider the opposite limit where even a single run of the experiment involves a large number of particles. This implies that a complete spatial interference pattern can be built up in a single run. In our special situation we will be able to give a full analytical characterization of these interference patterns. An alternative approach would be to interprete the correlation functions in terms of conditional probabilities. Numerical simulations of single atom counts, based on these conditional probabilities, would lead to a similar result.

In accordance with the experimental experience, we describe the density distribution $N(x)$, measured in a single run of the experiment, as a smooth pattern $\tilde{N}(x)$, blurred by some additional unpredictable and uncorrelated shot noise $\xi(x)$

$$N(x) = \tilde{N}(x) + \xi(x). \tag{2.13}$$

The shot noise accounts for the discreteness of the particles and is assumed to obey Poissonian statistics with

$$\begin{aligned} \Big\langle \xi(x) \Big\rangle &= 0 \\ \Big\langle \xi(x)\xi(x+\Delta) \Big\rangle &= \Big\langle N(x) \Big\rangle \delta(\Delta). \end{aligned} \tag{2.14}$$

Note, that $N(x)$ is a measured quantity and not a quantum mechanical operator or an ensemble average. This implies that both $\tilde{N}(x)$ and $\xi(x)$ may differ between individual runs of the experiment. Since the individual noisy contribution $\xi(x)$ is unpredictable, all we can hope to achieve, is a complete characterization of the smooth patterns $\tilde{N}(x)$.

It was indicated above, that an analysis of correlation functions will be required to make substantial statements about the behaviour of single runs. The lowest order correlation function is given by

$$\mathrm{corr}N(x, \Delta) = \Big\langle N(x)N(x+\Delta) \Big\rangle, \tag{2.15}$$

where the angular brackets denote an average over several runs of the experiment. We can calculate this correlation by identifying the measured density distribution $N(x)$ with the corresponding quantum mechanical operator $\Psi^\dagger(x)\Psi(x)$. This yields

$$\mathrm{corr}N(x, \Delta) = \Big\langle \Psi^\dagger(x)\Psi(x)\Psi^\dagger(x+\Delta)\Psi(x+\Delta) \Big\rangle. \tag{2.16}$$

Since we aim to characterize the smooth patterns $\tilde{N}(x)$, we are interested in correlations of $\tilde{N}(x)$, defined analogously to Eq. (2.15), instead of those of the measured density $N(x)$. Therefore, we rewrite Eq. (2.15) using Eq. (2.13) and Eq. (2.14) as

$$
\begin{aligned}
\mathrm{corr}N(x,\Delta) &= \mathrm{corr}\tilde{N}(x,\Delta) + \Big\langle \xi(x)\xi(x+\Delta) \Big\rangle \\
&= \mathrm{corr}\tilde{N}(x,\Delta) + \Big\langle N(x) \Big\rangle \delta(\Delta).
\end{aligned} \tag{2.17}
$$

Normal ordering of the field operators in Eq. (2.16) generates a term that is identical with the delta correlated noise in Eq. (2.17). The correlation function for the *smooth* pattern is thus given by

$$
\mathrm{corr}\tilde{N}(x,\Delta) = \Big\langle \Psi^\dagger(x)\Psi^\dagger(x+\Delta)\Psi(x+\Delta)\Psi(x) \Big\rangle. \tag{2.18}
$$

We will use this equation in the following to determine the distribution of all smooth patterns $\tilde{N}(x)$ that can be seen in single runs of the experiment.

Since the condensates are described as macroscopically occupied momentum modes, we have to link the above description in position space with one in momentum space. The momentum decomposition of the quantum mechanical field operator along the x-axis on the screen is

$$
\Psi(x) = \frac{1}{\sqrt{V^{1/3}}} \sum_k e^{ikx} a_k \tag{2.19}
$$

while any possible smooth interference pattern can be described by the Fourier decomposition

$$
\tilde{N}(x) = \frac{1}{V^{1/3}} \sum_k e^{ikx} \tilde{n}_k. \tag{2.20}
$$

Note that $\tilde{n}_k$ is a complex random variable with yet unknown statistics. It satisfies the relation $\tilde{n}_k = \tilde{n}^*_{-k}$ to ensure that $\tilde{N}(x)$ is real. In the following we will derive the statistics of the Fourier components $\tilde{n}_k$ by a comparison of the higher moments of $\tilde{n}_k$ with higher moments of the quantum mechanical operators a_k.

Identification of the average over the measured screen patterns $\Big\langle \tilde{N}(x) \Big\rangle$ with the quantum mechanical prediction $\Big\langle \Psi(x)^\dagger \Psi(x) \Big\rangle$

$$
\frac{1}{V^{1/3}} \sum_k e^{ikx} \langle \tilde{n}_k \rangle = \frac{1}{V^{1/3}} \sum_k e^{ikx} \sum_{k'} \Big\langle a^\dagger_{k'-k} a_{k'} \Big\rangle \tag{2.21}
$$

yields

$$
\langle \tilde{n}_k \rangle = \sum_{k'} \Big\langle a^\dagger_{k'-k} a_{k'} \Big\rangle \tag{2.22}
$$

due to the uniqueness of the Fourier decomposition. Since the two momentum modes are assumed to be independent we get

$$
\langle \tilde{n}_k \rangle = \delta_{k,0} \left(N_1 + N_2 \right). \tag{2.23}
$$

Note, that the *ensemble* averages $\langle a_k \rangle$ vanish. In a similar way the correlation of the measured patterns $\mathrm{corr}\tilde{N}$

$$\mathrm{corr}\tilde{N}(x,\Delta)V^{2/3} = \sum_{kk'} e^{i(k+k')x} e^{ik\Delta} \langle \tilde{n}_k \tilde{n}_{k'} \rangle \tag{2.24}$$

can be identified with its quantum mechanical analogon

$$\begin{aligned}\mathrm{corr}\tilde{N}(x,\Delta)V^{2/3} = &\left\langle a_1^\dagger a_1^\dagger a_1 a_1 \right\rangle + \left\langle a_2^\dagger a_2^\dagger a_2 a_2 \right\rangle \\ &+2N_1 N_2 \left(1 + \cos 2k_0 \Delta\right).\end{aligned} \tag{2.25}$$

Eq. (2.25) states that the probability for observing an atom at a position $x + \Delta$ after having observed a previous atom at position x is a periodic function of $2k_0\Delta$. Already this observation implies that the density distribution of a single run must show some spatial variation. Eq. (2.24) and Eq. (2.25) determine the only nonvanishing coefficients $\langle \tilde{n}_k \tilde{n}_{k'} \rangle$

$$\left\langle \tilde{n}_0^2 \right\rangle = \left\langle a_1^\dagger a_1^\dagger a_1 a_1 \right\rangle + \left\langle a_2^\dagger a_2^\dagger a_2 a_2 \right\rangle + 2N_1 N_2, \tag{2.26}$$

$$\left\langle |\tilde{n}_{\pm 2k_0}|^2 \right\rangle = N_1 N_2. \tag{2.27}$$

In the case of *small number fluctuations*, as assumed in Eq. (2.8), we get

$$\left\langle \tilde{n}_0^2 \right\rangle = (N_1 + N_2)^2 = \langle \tilde{n}_0 \rangle^2. \tag{2.28}$$

This equality implies that $\tilde{n}_0$ does not fluctuate. Rather than being a random variable, its value is invariably given by

$$\tilde{n}_0 = N_1 + N_2. \tag{2.29}$$

Eq. (2.27) and the vanishing expectation value of $\tilde{n}_{2k_0}$ Eq. (2.23) show that at least the phase of the Fourier coefficient $\tilde{n}_{2k_0}$ is subject to fluctuations. A similar evaluation of the fourth order correlation function yields

$$\left\langle |\tilde{n}_{2k_0}|^4 \right\rangle = (N_1 N_2)^2. \tag{2.30}$$

We skipped the explicit calculation here. One sees by a comparison with Eq. (2.27) that the *modulus* of $\tilde{n}_{2k_0}$ does not fluctuate either. It is fixed to the value

$$|\tilde{n}_{2k_0}| = \sqrt{N_1 N_2}. \tag{2.31}$$

Since the classical correlation functions are of even order in the atomic field operators, any phase factors cancel and no constraint for the phase of $\tilde{n}_{2k_0}$ exists.

Thus, Eq. (2.29) and Eq. (2.31) provide us with the complete obtainable information about the Fourier coefficients of the interference patterns (cf. Eq. (2.20)) that can be measured in single runs of the experiment. Since no constraint for the phase of $\tilde{n}_{2k_0}$ exists, its value φ is an equally distributed random variable. The result of this derivation is, that the distribution of all possible recorded *smooth* interference patterns $\tilde{N}(x)$, rather than the full noisy patterns $N(x)$, is given by Eq. (2.11).

A surprising consequence is that with Eq. (2.11) holds (up to order $1/N_i$) no matter whether the density matrices of the individual modes are given by Eq. (2.6) or Eq. (2.7). This shows that a Fock state will yield in the limit of large particle numbers virtually the

same experimental results as an incoherent superposition of coherent states (see also Ref. [6]). The neglected terms of order $1/N_i$ mainly account for the existence of some correlations in the earlier eliminated noise $\xi(x)$ in order to preserve the total particle number precisely.

We close this section with a few remarks about the range of validity of the above result under more realistic conditions. We have assumed in our calculations that the fluctuations in the flux are negligible during the detection time τ_d. Inclusion of these fluctuations will clarify the range of validity of the above considerations. When taking into account that the flux of particles may vary during the formation of the interference pattern, the correlation function Eq. (2.15) changes to

$$\mathrm{corr}N(x,t;\Delta,\tau) = \Big\langle N(x,t)N(x+\Delta,t+\tau)\Big\rangle. \tag{2.32}$$

Eq. (2.25) can then be rewritten in terms of the time correlation functions that characterize the dynamics of the two independent modes ($i = 1, 2$)

$$\begin{aligned}\mathrm{corr}\tilde{N}(\Delta,\tau)V^{2/3} = & N_1^2 g_1^{(2)}(\tau) + N_2^2 g_2^{(2)}(\tau) \\ & +2N_1N_2\left(1+\cos 2k_0\Delta Re\left\{g_1^{(1)}(\tau)g_2^{(1)*}(\tau)\right\}\right)\end{aligned} \tag{2.33}$$

with

$$\begin{aligned} g_i^{(1)}(\tau) &= \frac{\Big\langle a^\dagger(t)a(t+\tau)\Big\rangle}{\sqrt{\langle a^\dagger(t)a(t)\rangle\,\langle a^\dagger(t+\tau)a(t+\tau)\rangle}} \overset{\tau\to\infty}{\longrightarrow} 0 \\ g_i^{(2)}(\tau) &= \frac{\Big\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\Big\rangle}{\langle a^\dagger(t)a(t)\rangle\,\langle a^\dagger(t+\tau)a(t+\tau)\rangle} \overset{\tau\to\infty}{\longrightarrow} 1. \end{aligned} \tag{2.34}$$

It is evident that the decay of the single interference pattern during a longer exposure time is intimately connected to the decay of the amplitude correlation function $g^{(1)}(\tau)$. Such a decay will happen in any finite physical system. $g^{(2)}(\tau)$ will decay on a similar time scale and we get

$$\lim_{\tau\to\infty} \mathrm{corr}\tilde{N} = (N_1+N_2)^2\,. \tag{2.35}$$

This implies that even a single run is appropriately described by the quantum mechanical ensemble average $\Big\langle \Psi^\dagger(x)\Psi(x)\Big\rangle$, as we are used from ordinary atomic interference experiments. However, in a system with a strong quantum degeneracy like a laser or a Bose condensate the decay of $g^{(1)}(\tau)$ may be so slow compared to the flux of atoms so that intermediate time scales are relevant for experimental purposes. The concept of broken symmetry applies only for these time scales.

III. INTERFERENCE OF TWO LOCALIZED CONDENSATES

In the last section we approximated the condensates by two plane waves. The general result, however, is equally valid for more realistic setups. Here we show by a numerical simulation that the setup of [3] indeed implements such an interference experiment, although at present no interference has been observed due to an insufficient experimental resolution.

The experiment, reported in [3], can be summarized as follows. Two independent Bose condensates of ^{23}Na atoms are stored in a magnetic trap, separated by a laser beam. These constraints can be modelled by two identical harmonic potentials with angular frequencies $\nu = 2\pi \times 745 s^{-1}$, $2\pi \times 235 s^{-1}$, $2\pi \times 410 s^{-1}$ in the x, y, and z directions, respectively. These potentials are separated along the x-axis by $x_0 = 100\mu$m. The total number of atoms is assumed to be $N_0 = 1.5 \times 10^5$, or 0.75×10^5 atoms in each condensate. The scattering length, characterizing the atomic interaction, amounts to $a = 4.9$nm. After switching off the constraints, the atomic clouds expand and eventually overlap, both due to the quantum mechanical spreading of their wave packets and to the atomic repulsion. The final spatial distribution of the atoms is measured after 6ms or equivalently $28\nu_x^{-1}$.

In our calculation we simulate the expansion of these atomic clouds. We assume that the two condensates initially are at zero temperature, with both condensates equally populated. The investigation of atomic interference at nonvanishing temperatures is put forward to the next section. There, it is shown that interference can be observed in a different setup also at small but finite temperatures.

The sodium atoms are subject to a repulsive interaction. The initial wave function of a single condensate is therefore described by a solution of the Gross-Pitaevskii equation [8]

$$\mu\psi(\vec{r},t) = \left(-\frac{\hbar^2 \vec{\nabla}^2}{2m} + V(\vec{r}) + \tilde{U}|\psi(\vec{r},t)|^2 \right) \psi(\vec{r},t) \tag{3.1}$$

with the chemical potential μ and

$$\tilde{U} = N_0 \frac{4\pi\hbar^2 a}{m}. \tag{3.2}$$

Note that the macroscopic wave function $\psi(\vec{r},t)$ is normalized to one in this notation. The chemical potential

$$\mu = 0.118 \frac{\tilde{U}}{N_0} \left(\frac{N_0}{\Delta x^2 \Delta y^2 \Delta z^2 a^{3/2}} \right)^{2/5} \tag{3.3}$$

is expressed in terms of the natural length units of the harmonic potential $V(\vec{r})$ along the coordinate axes. In the case of the x-direction, for example, this length unit is

$$\Delta x = \sqrt{\frac{\hbar}{m\nu_x}} = 0.78\mu\text{m}. \tag{3.4}$$

Since the total energy is dominated by the particle interactions and the external harmonic potential, it is a good first approximation [16] to neglect the kinetic energy in Eq. (3.1). This yields the approximate expression

$$\psi(\vec{r}) \approx \sqrt{\left(\mu - V(\vec{r})\right)/\tilde{U}} \tag{3.5}$$

for the initial wave function, provided that the expression under the square root is positive. Otherwise, the wave function vanishes abruptly, for example on the x-axis at a distance of 3.24μm from the origin. The physical wave function, however, vanishes smoothly on a

small scale that is called the healing length [16]. This smooth rather than abrupt vanishing accounts for the existence of a limited kinetic energy.

We therefore conclude that the initial wave function of a single condensate is largely of a parabolic shape and about 4 times larger than the corresponding noninteracting wave function. Note, that this significant deviation from the pure harmonic oscillator ground state is due to a balance between the potential energies of the atom interactions and the external harmonic potential. It should not be confused with the depletion of the condensate that is found in an interacting homogenous gas [8]:

$$\frac{N - N_0}{N} = \frac{8}{3}\sqrt{\frac{na^3}{\pi}}. \tag{3.6}$$

Inserting the peak densities of the current experiment into this formula indicates a depletion of less than 1% which is negligible for our purposes.

According to the results of the last section it is justified to describe the two independent condensates by one combined macroscopic wave function, for characterizing a single run of the experiment. This wave function consists of two spatially separated parts, corresponding to the two condensates, that are joined with a relative phase φ between them. Of course, this relative phase varies between different runs of the experiment. Since variations in the relative phase lead only to trivial shifts of the interference pattern we restrict ourselves to the case $\varphi = 0$.

The trap is switched off at a given time and the atomic clouds expand. We describe this expansion assuming the validity of the nonlinear Schrödinger equation [16]

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = \left(-\frac{\hbar^2\vec{\nabla}^2}{2m} + \tilde{U}|\psi(\vec{r},t)|^2\right)\psi(\vec{r},t). \tag{3.7}$$

This is also referred to as the time-dependent Gross-Pitaevskii equation (cf. Eq. (3.1)). Its formal structure resembles a mean-field equation when replacing the original two-body potential by a pseudopotential. It can be derived for cold and dilute gases by applying the ladder approximation to the two-body Green's function [8]. Even within this approximation Eq. (3.7) does not account for the full dynamics, but it should be a good approximation for time scales, where collisional rates can be neglected.[1]

Due to the nonlinearity in Eq. (3.7), it is incorrect to separate the wave function with respect to the spatial coordinates. Nevertheless, a separation of the spatial degrees of freedom has to be applied as an approximation, since a full three-dimensional numerical treatment of the problem is far beyond reach. Such an approximation goes back to [17] and has found widespread use in chemical physics where it is called time-dependent self-consistent field (TDSCF) method. Its validity has been studied for example in [18]. This approximation reduces the propagation of a three-dimensional wave function to that of three coupled one-dimensional wave functions. The error of this simplification depends on the deviation of the wave function from a Gaussian distribution.

[1]Note, that the atomic clouds are not in thermal equilibrium during the described expansion. The earlier assumption of zero temperature applies only in the presence of the trapping potentials.

The numerical propagation of the wave function is performed by a split operator technique [19] that is accurate and easy to implement. The initial wave function is generated by an imaginary time propagation of Eq. (3.7), which yields the full solution of Eq. (3.1) in contrast to the approximate wave function Eq. (3.5).

We plot in Fig. 1 the time evolution of the x distribution, as it could have be seen in the experiment. We have averaged over the relative phase of the two condensates since the fringes are not resolvable on the plotted scale. The two wave functions overlap after about 3ms. Even though there is a considerable overlap of the two condensates, it is not large enough to allow for an interpretation of the x distribution in terms of a velocity distribution of the initial single condensates.

In Fig. 2 we show both the initial and final x distribution including the interference fringes as they are predicted by the nonlinear Schrödinger equation. The periodic length of the fringes is about 1μm and thus below the pixel resolution of 12.4μm used in the experiment of [3]. The period of the fringes increases slowly with time. However, its size is of the same order of magnitude as the healing length throughout the whole investigated time interval. The period of the fringes could be enlarged by widening the trap potentials before dropping the atoms. This would reduce both the initial kinetic and interaction energy.

Fig. 3 shows the calculated x-z distribution after 6ms. Its square like shape is an artifact of the TDSCF approximation. A full three-dimensional calculation would yield a slightly larger extension along the axes. For comparison, we plot in Fig. 4 the corresponding distribution, calculated from the experimental data of [3]. The original data describe the spatial dependence of the light absorption. Assuming an exponential absorption law we have transformed the light absorption into a particle distribution and normalized it for a better comparison with Fig. 3. We conclude that the experimental data of [3] are in good agreement with a description using the nonlinear Schrödinger equation. While the existence of shoulders and their positions are in especially good agreement, also the total size of the atomic cloud is fairly well reproduced. A detailed quantitative analysis, however, requires a better accuracy of the experimental data.

IV. INTERFERENCE PROPERTIES OF A SINGLE BOSE CONDENSATE

In this Section we analyze the effects of finite temperatures in interference experiments with a Bose gas. We consider a sample of N bosons confined in a three-dimensional harmonic trap, not necessarily isotropic. After reaching thermal equilibrium at a given temperature T, the gas is released and the sample falls through two pinholes, separated by a distance $2d$, that are located on a screen S_1 at a distance L from the trap center (Fig. 5). The atoms are recorded on a second screen S_2, at a distance D from the trap center. We wish to stress the fact that in this experiment the wavefunctions coming from both pinholes have fixed relative phases since they represent filtered parts from the same large wavefunction that was stored in the trap. Therefore, different realizations of the experiment will show the same interference pattern. This is in contrast with the experiments analyzed in Sec. II, in which we considered two *independent* condensates. We will not perform the correlation analysis of Sec. II here since it does not yield additional information in this case. The main purpose of this Section is to show how the interference pattern depends on the temperature of the sample, and to what extent it reflects the phase transition at the critical temperature. We

will consider the case of an ideal gas, i.e. we will not take into account the role of atomic collisions. The problem of a (weakly) interacting Bose gas at finite temperatures deserves a separate analysis.

A. Evolution through the pinholes

We are interested in the mean number of particles dN per unit time and unit area deposited at any point $\vec{r} = (x, y, D)$ on the screen S_2 at a given time t. This quantity is given by the expectation value of the z component of the probability current,

$$I(\vec{r}, t) \equiv \frac{dN}{dS\ dt} = \frac{\hbar}{M} \mathrm{Im} \langle \Psi(\vec{r}, t)^\dagger \frac{\partial}{\partial z} \Psi(\vec{r}, t) \rangle, \tag{4.1}$$

where M is the atomic mass, and $\Psi(\vec{r}, t)$ is the field operator describing the sample. For the case of non–interacting particles, this operator can always be written as

$$\Psi(\vec{r}, t) = \sum_{\vec{n}} \hat{a}_{\vec{n}} \psi_{\vec{n}}(\vec{r}, t), \tag{4.2}$$

where the wavefunction $\psi_{\vec{n}}(\vec{r}, t)$ satisfies the (single particle) Schrödinger equation describing the evolution of a single particle. Here, $\psi_{\vec{n}}(\vec{r}, 0)$ is the eigenfunction of the three-dimensional harmonic oscillator with quantum numbers $\vec{n} = (n_x, n_y, n_z)$ $(n = 0, 1, \ldots)$. As usual, $a_{\vec{n}}$ are annihilation opeartors of particles in the state $\vec{n}$.

It is worth mentioning that, if the wavefunction is expanded in terms of momentum eigenstates as in the previous Section, $I(\vec{r}, t)$ can be fully characterized by the density of particles. Here, however, we find it more convenient to use the eigenstates of the harmonic oscillator. Therefore the probability current has to be evaluated explicitly for describing the interference pattern.

Substituting (4.2) into (4.1), we obtain

$$I(\vec{r}, t) = \sum_{\vec{n}} N_{\vec{n}} I_{\vec{n}}(\vec{r}, t) \tag{4.3}$$

implying that the incoherently populated trap eigenfunctions propagate independently after releasing the trap. Their occupation numbers $N_{\vec{n}} = \langle a_{\vec{n}}^\dagger a_{\vec{n}} \rangle$ are given by the Bose–Einstein distribution

$$N_{\vec{n}} = \frac{\lambda e^{-\beta\hbar(n_x\nu_x + n_y\nu_y + n_z\nu_z)}}{1 - \lambda e^{-\beta\hbar(n_x\nu_x + n_y\nu_y + n_z\nu_z)}} \tag{4.4}$$

where $\beta = 1/\kappa_B T$, λ is related to the number of particles in the trap N, and $\nu_{x,y,z}$ the trap frequencies along the three axes (see Appendix B). We have also defined

$$I_{\vec{n}}(\vec{r}, t) = \frac{\hbar}{M} \mathrm{Im} \left[\psi_{\vec{n}}(\vec{r}, t)^* \frac{\partial}{\partial z} \psi_{\vec{n}}(\vec{r}, t) \right]. \tag{4.5}$$

In view of this formula, we only need to derive the expression for $\psi_{\vec{n}}(\vec{r}, t)$; that is, we simply have to find the evolution through the pinholes of a single particle that is initially in the harmonic oscillator eigenstate with quantum numbers $\vec{n}$.

Apart from neglecting the role of collisions, in order to calculate the evolution of $\psi_{\vec{n}}(\vec{r},0)$ we will make other simplifying assumptions. First, we will assume that L is small enough such that during the time that the sample needs to reach the pinholes, this wavefunction basically does not change. Second, we will assume that the size of the pinholes is small compared to the typical distances over which this wavefunction varies. Finally, we will take D to be large enough in such a way that the parts of the wavefunction coming from each pinhole overlaps with each other. According to these assumptions, we can approximate

$$\psi_{\vec{n}}(\vec{r},0) \simeq k\psi_{n_x}(x)[\delta(x-x_1)+\delta(x-x_2)]\psi_{n_y}(y)\delta(y-y_0)\psi_{n_z}(z), \tag{4.6}$$

where (x_1, y_0) and (x_2, y_0) are the coordinates of the pinholes in the first screen S_1, $\psi_n(x)$ is the eigenstate of the one-dimensional harmonic oscillator in position representation, and k a normalization constant.

After a time $t > 0$, the wavefunction evolves to

$$\psi_{\vec{n}}(\vec{r},t) \simeq k[G_0(x,x_1,t)\psi_{n_x}(x_1)+G_0(x,x_2,t)\psi_{n_x}(x_2)]G_0(y,y_0,t)\psi_{n_y}(y_0)\psi_{n_z}(z,t), \tag{4.7}$$

where

$$\psi_{n_z}(z,t) = \int_{-\infty}^{\infty} dz' G_g(z,z',t)\psi_{n_z}(z'). \tag{4.8}$$

Here,

$$G_0(x,x',t) = e^{-i\pi/4}\left[\frac{M}{2\pi\hbar t}\right]^{1/2} e^{i\frac{M(x-x')^2}{2\hbar t}} \tag{4.9a}$$

$$G_g(z,z',t) = e^{-i\pi/4}\left[\frac{M}{2\pi\hbar t}\right]^{1/2} e^{i\frac{M}{2\hbar t}[(z-z')^2+gt^2(z+z')-g^2t^3/12]} \tag{4.9b}$$

are the free propagator and the propagator under a constant force, respectively. The substitution of (4.7) into (4.5) allows us to write

$$I_{\vec{n}}(\vec{r},t) = k^2 I^x_{n_x}(x,t) I^y_{n_y}(y,t) I^z_{n_z}(z,t) \tag{4.10}$$

where

$$I^x_{n_x}(x,t) = |G_0(x,x_1,t)\psi_{n_x}(x_1)+G_0(x,x_2,t)\psi_{n_x}(x_2)|^2, \tag{4.11a}$$

$$I^y_{n_y}(y,t) = |G_0(y,y_1,t)\psi_{n_y}(y_0)|^2, \tag{4.11b}$$

$$I^z_{n_z}(z,t) = \frac{\hbar}{M}\mathrm{Im}\left[\psi_{n_z}(z,t)^*\frac{\partial}{\partial z}\psi_{n_z}(z,t)\right]. \tag{4.11c}$$

Let us now derive simple expressions for these quantities. First, using (4.9a) we find

$$I^y_{n_y}(y,t) = (M/2\pi\hbar t)|\psi_{n_y}(y_0)|^2 \tag{4.12}$$

for all y [20]. On the other hand, as it is shown in the Appendix A,

$$I^z_{n_z}(z,t) = F(t)|\tilde{\psi}_{n_z}(z-gt^2/2,0)|^2, \tag{4.13}$$

where

$$F(t) = \frac{gt}{1+(\nu_z t)^2}\left(1+\frac{\nu_z^2}{g}\left[\frac{1}{2}gt^2 - z\right]\right), \tag{4.14}$$

and $\tilde{\psi}_{n_z}$ is the eigenstate of the one-dimensional harmonic oscillator with frequency $\nu_z/[1+(\nu_z t)^2]$. Finally,

$$I^x_{n_x}(x,t) = (M/2\pi\hbar t)\{|\psi_{n_x}(x_1)|^2 + |\psi_{n_x}(x_2)|^2 + 2\psi_{n_x}(x_1)\psi_{n_x}(x_2)\cos[2\pi x/x_f(t) + \phi(t)]\} \tag{4.15}$$

where we have taken into account that $\psi_{n_x}(x)$ is real [20]. Here,

$$x_f(t) = \frac{\pi\hbar t}{Md} = 2\pi\nu_x t\frac{a_{0x}^2}{d} \tag{4.16}$$

with $a_{0x} = [\hbar/(2M\nu_x)]^{1/2}$ being the size of the ground state wavefunction, and $\phi(t) = M(x_1^2 - x_2^2)/(2\hbar t)$.

B. Interference and visibility

Let us now specialize the above derived expressions to a simple case. We take $y_0 = 0$ and $x_1 = d = -x_2$ (i.e. the pinholes are symmetrically situated along the x axis). For the sake of simplicity we choose the time $\tau = (2D/g)^{1/2}$, which is the time required for the center of the trap to reach by free fall the screen S_2. In this case, we obtain

$$I^x_{n_x}(x,\tau) \propto |\psi_{n_x}(d)|^2\{1+(-1)^{n_x}\cos[2\pi x/x_f(\tau)]\}, \tag{4.17a}$$
$$I^y_{n_y}(y,\tau) \propto |\psi_{n_y}(0)|^2, \tag{4.17b}$$
$$I^z_{n_z}(z,\tau) \propto |\psi_{n_z}(0)|^2, \tag{4.17c}$$

Note that the term $(-1)^{n_x}$ appearing in the first of these expresions comes from the fact that $\psi_{n_x}(-d) = (-1)^{n_x}\psi_{n_x}(d)$. Thus, depending on n_x being even or odd, the interference pattern has a maximum or a minimum at $x = 0$. It is obvious that the incoherent addition of all the contributions for different quantum numbers n_x [see (4.3)] will decrease the visibility. Therefore, the origin of the temperature dependence of the interference fringes is precisely the term $(-1)^{n_x}$ accompanying the cosine. For sufficiently low temperature, the Bose–Einstein distribution (4.4) is peaked at $n_x = 0$, and therefore fringes along the y axis will be clearly displayed on the screen S_2. The separation of the fringes is $x_f(\tau)$ given in (4.16). Note that, in principle, this separation changes with time. In practice, this variation will be negligible as long as the time required by the sample to cross the screen $\delta\tau$ is much smaller than τ. More specifically, we can neglect this time dependence for

$$\frac{\delta\tau}{\tau} = \sqrt{n_z[1+(\nu\tau)^2]}\frac{a_{0z}}{2D} \ll 1. \tag{4.18}$$

This condition can always be fulfilled for a sufficiently long distance D (note that τ scales as $\sqrt{D}$, and therefore $\delta\tau/\tau$ scales as $1/\sqrt{D}$).

Let us now analyze the temperature dependence of the visibility

$$V = \frac{I^{\max} - I^{\min}}{I^{\max} + I^{\min}}. \tag{4.19}$$

where I is given in (4.3). The maxima and minima of the expression (4.17a) can be easily calculated, taking into account that for fixed $n_{y,z}$, $N_{\vec{n}}$ is a decreasing function of n_x. Thus, the maxima (minima) in the fringes correspond to $x_k^{\max} = kx_f(\tau)$ [$x_k^{\min} = (k+1/2)x_f(\tau)$], with $k = 0, \pm 1, \ldots$. Using this result we find that

$$I^{\max} + I^{\min} \propto \sum_{\vec{n}} N_{\vec{n}} |\psi_{n_x}(d)|^2 |\psi_{n_y}(0)|^2 |\psi_{n_z}(0)|^2, \tag{4.20a}$$

$$I^{\max} - I^{\min} \propto \sum_{\vec{n}} (-1)^{n_x} N_{\vec{n}} |\psi_{n_x}(d)|^2 |\psi_{n_y}(0)|^2 |\psi_{n_z}(0)|^2. \tag{4.20b}$$

In Fig. 6 we have plotted the visibility (solid–lines) as a function of the scaled temperature $\kappa_B T/(\hbar\nu)$ for $N = 10000$ particles in an isotropic harmonic oscillator and different distances $2d$ between the two pinholes. We have also plotted (dotted–line) the proportion of particles in the condensate N_0 (for the numerical method used to calculate these results see Appendix B). Note first that for a given temperature, the visibility decreases as d increases. The reason for that is as follows. As mentioned above, the eigenstates of the harmonic oscillator with even quantum numbers n_x give *in phase* contributions to the fringe pattern, whereas the ones with odd n_x give *out of phase* contributions, i.e. tend to decrease the visibility. The effect of the terms with odd n_x tends to zero for $d \to 0$, since $\psi_{n_x}(d)$ tends also to zero (note that this wavefunction is antisymmetric). Thus, the pinholes select the particles whose wavefunctions have even n_x. As d increases, up to the order of the ground state wavefunction a_{0x}, the visibility decreases since the contributions of the odd wavefunctions becomes more important. For $d \gg a_{0x}$ the visibility remains small even below the critical point, since only the wavefunctions with high n_x contribute. On the other hand, the transition point of BEC is reflected in the visibility, which shows that double–slit experiments can be used to observe the phase transition. Note the dramatic changes at the transition point for moderate values of d. It is somehow surprising that even for $N - N_0 > N_0$ the visibility already becomes very large, since one would expect that the particles that are out of the condensate would produce some noise that would dominate over the effect of the condensed particles. However, this is not the case. The $N - N_0$ particles out of the condensate are distributed among several states with different n_x (this distribution is similar to a Boltzmann distribution and therefore has a long tail). As n_x increases, their contribution to the intensity becomes smaller and smaller given that $|\psi_{n_x}(d)|^2 \to 0$. Thus, for $d \lesssim a_{0x}$ the change of visibility at the transition point becomes very pronounced.

V. CONCLUSION

We have shown that two independent Bose condensates, when overlapping on a screen, can exhibit an interference pattern in a single run of the experiment. A rigorous analysis in terms of correlation functions confirmed that the description of an individual Bose condensate by a coherent state, i.e. a macroscopic wave function, gives the proper characterization of single runs of this interference experiment. This applies although atomic coherent states

violate fundamental conservation laws and holds in the limit of large occupation numbers even for a single Fock state. The macroscopic wave function concept resembles the way Quantum Electrodynamics converges to Classical Electrodynamics for large photon numbers. However, one fundamental difference between atomic and photonic coherent states should be mentioned. It is, at least in principle, possible to measure the absolute phase of a photon coherent state, since the electric field amplitude is a true observable. As a consequence, the photon number is no conserved quantity. In contrast, the atom number is strictly conserved. In return, the macroscopic wave function itself is not an observable, thus inhibiting the measurement of its *absolute* phase. Consequently, the investigated interference experiment yielded only information about the *relative* phase between two macroscopic wave functions.

It is another aim of the paper to describe simple experiments that allow for a comparison between experimental and theoretical features of a Bose condensate. The considerations above, concerning the interference of independent Bose condensates allowed the numerical simulation of a recent experiment [3] where two independent Bose condensates are released from a magnetic trap. They expand and eventually overlap. In our calculations we accounted for particle interactions but assumed an initial distribution with zero temperature. The calculations predict interference fringes with a period below the resolution of the current experimental data. However, the size of the final atomic cloud is in good agreement with the experimental result. A further quantitative analysis is limited by the present accuracy of the experiment. By improved detection techniques it should be possible to study the existence of the interference fringes. This could yield substantial and new information about the condensate and the range of validity of the nonlinear Schrödinger equation.

As a possible future experiment we proposed the study of the interference properties of a Bose condensed system in a conventional Young interference experiment. We assumed that the atoms are released from a harmonic trap and fall freely through two pinholes. Here we neglected particle interactions since the above calculations have shown that the visibility of the interference fringes is not severely affected by these interactions. Instead we studied the dependence of the visibility on the temperature. The transition to a Bose condensed phase is reflected by a sharp rise in the visibility of the fringes. Under favourable conditions the visibility reaches values close to one immediately below the critical temperature. In addition a variation of the distance of the the pinholes shows that the coherence length of a finite Bose condensed system is comparable with the size of the groundstate wavefunction.

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APPENDIX A: EVOLUTION IN THE Z DIRECTION

In this appendix we derive some of the formulas used in Section III. First, starting from (4.8) we write

$$\psi_n(z,t) = A(z,t) K_n \int_{-\infty}^{\infty} dz' H_n(\alpha z') e^{-R(t) z'^2 + i C(z,t) z'}. \tag{A1}$$

Here, we have used

$$\psi_n(z,0) = K_n H_n(\alpha z) e^{-\alpha^2 z^2/2}, \tag{A2}$$

where $\alpha = (M\nu_z/\hbar)^{1/2}$, $K_n = [\alpha/(2^n n! \sqrt{\pi})]^{1/2}$, and H_n is the n–th Hermite polynomial. We have also defined

$$A(z,t) = e^{-i\pi/4} \sqrt{\frac{M}{2\pi\hbar t}} e^{iM/(2\hbar t)(z^2 + g t^2 z - g^2 t^4/12)}, \tag{A3a}$$

$$C(z,t) = \frac{M}{\hbar t}\left[\frac{1}{2} g t^2 - z\right], \tag{A3b}$$

$$R(t) = \frac{\alpha^2}{2} - i\frac{M}{2\hbar t} = \frac{M}{2\hbar t}(\nu_z t - i). \tag{A3c}$$

Performing the change of variables $\sqrt{R(t)} z = x$ and transforming the path for the integration in the complex plane, we arrive at

$$\psi_n(z,t) = \frac{A(z,t)}{\sqrt{R(t)}} K_n e^{-B(z,t)^2} \int_{-\infty}^{\infty} dz' H_n\left[\frac{\alpha z'}{\sqrt{R(t)}}\right] e^{-[x - iB(z,t)]^2}, \tag{A4}$$

where $B(z,t) = C(z,t)/[2\sqrt{R(t)}]$. Performing the integration, and after some lenghty algebra we obtain

$$\psi_n(z,t) = \frac{A(z,t)}{\sqrt{R(t)}} K_n e^{-\tilde{\alpha}(t)^2 \tilde{Z}(z,t)^2/2[1 + i/(\nu_z t)]} [-R(t)^*/R(t)]^{n/2} H_n[\tilde{\alpha}(t)\tilde{Z}(z,t)], \tag{A5}$$

where $\tilde{\alpha} = \alpha/\sqrt{1+\nu_z^2 t^2}$ and $Z(z,t) = z - gt^2/2$. Now, the expression (4.13) can be easily derived starting from (4.11c) and using (A5). To do that, one has to note that when performing the derivatives of $\psi_n(z,t)$, the contribution given by the derivative of the Hermite polynomial vanishes when taking the imaginary part. We obtain

$$I^z_{n_z}(z,t) = F(t) |\psi_{n_z}(z,t)|^2, \tag{A6}$$

where $F(t)$ is given in (4.14). Taking the modulus square of (A5) we obtain (4.11c).

APPENDIX B: FORMULAS FOR THE NUMERICAL EVALUATION OF THE VISIBILITY

In this appendix we give some of the formulas used to evaluate numerically the expression of the visibility for a Bose–Einstein distribution. The total number of particles can be written as

$$N = \sum_{\vec{n}} N_{\vec{n}} = \sum_{k=1}^{\infty} \left[\frac{N_0}{N_0+1}\right]^k \prod_{i=x,y,z} (1 - e^{-\beta\hbar k \nu_i})^{-1}. \tag{B1}$$

Given a fixed number of particles N and a temperature T, we have first determined the value of N_0 by using a bisection method varying N_0 until Eq. (B1) is verified. Thus, we can calculate the whole distribution through Eq. (4.4) since $\lambda = N_0/(1+N_0)$.

On the other hand, in order to calculate the expressions (4.20) we need to determine sums of the form

$$S_1 = \sum_{n=0}^{\infty} e^{-\beta\hbar\nu k(n+1/2)} |\psi_n(x)|^2, \tag{B2a}$$

$$S_2 = \sum_{n=0}^{\infty} (-1)^n e^{-\beta\hbar\nu k(n+1/2)} |\psi_n(x)|^2. \tag{B2b}$$

This can be easily performed by noting that these expressions are related to the propagator for the harmonic oscillator with an imaginary time. In particular,

$$S_1 = G(x, x, t = -i\beta k) \tag{B3a}$$

$$S_2 = G[x, x, t = \pi/(\hbar\nu) - i\beta k], \tag{B3b}$$

where

$$G(x, x, t) = \sqrt{A/(2\pi)} e^{Ax^2[1-\cos(\nu t)]}, \tag{B4}$$

with $A = M\nu/[i\hbar \sin(\nu t)]$. Using these expresions, we find

$$I^{\max} + I^{\min} = \sum_{k=1}^{\infty} \left[\frac{N_0}{N_0+1}\right]^k e^{\alpha^2 d^2 [1-\cosh(\beta\hbar\nu_x k)]/\sinh(\beta\hbar\nu_x k)} \prod_{i=x,y,z} \sqrt{\sinh(\beta\hbar\nu_i k)}, \tag{B5a}$$

$$I^{\max} - I^{\min} = \sum_{k=1}^{\infty} \left[\frac{N_0}{N_0+1}\right]^k e^{-\alpha^2 d^2 [1+\cosh(\beta\hbar\nu_x k)]/\sinh(\beta\hbar\nu_x k)} \prod_{i=x,y,z} \sqrt{\sinh(\beta\hbar\nu_i k)}, \tag{B5b}$$

(B5c)

Note that one could use directly the formulas (4.20), which involve 3 nested sums. However, Eq. (B5) involves a single sum, which saves a lot of computer time.

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* Also at Sektion Physik, Universität München, Theresienstraße 37, D–80333 München, Germany.

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[20] Given that we have approximated the initial state by a delta function, this result involves a infinitely fast spreading of the wavefunction. In reality, the results will be valid over a finite region of the screen S_2 whose dimensions will increase with the distance D.

FIGURES

FIG. 1. Calculated time evolution of the x distribution. The interference fringes are removed by averaging over φ. The contour lines correspond to an increment of 0.001.

FIG. 2. Initial and final x distribution. The final distribution is plotted both with a relative phase $\varphi = 0$ and averaged over all possible phases.

FIG. 3. Calculated final x-z distribution with removed interference fringes.

FIG. 4. Experimental atomic density, calculated from the light absorption. A renormalization was performed to facilitate comparison with Fig. 3. The original data correspond to Fig. 2c of [3]. Their use is with kind permission of W. Ketterle and coworkers.

FIG. 5. Setup of the experiment considered in Section III. An ideal Bose gas in thermal equilibrium at temperature T is dropped through two pinholes. The interference is recorded on a screen.

FIG. 6. Visibility V plotted as a function of the scaled temperature for (curves from top to bottom) $d = 0.5, 1.56, 2.61, 3.67, 4.72, 5.78,$ and $6.83[\hbar/(M\nu)]$ (the inset shows a detail). In dotted line we have plotted the proportion of particles in the condensate $n_0 = N_0/N$.